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**The Governing Macroscopic Equations  
of Partially Ionized Gases**

**25 JULY 1962**

*Prepared by* **D. M. DIX**  
**Physical Research Laboratory**

*Prepared for* **DEPUTY COMMANDER AEROSPACE SYSTEMS**

**AIR FORCE SYSTEMS COMMAND**

**UNITED STATES AIR FORCE**

*Inglewood, California*

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THE GOVERNING MACROSCOPIC EQUATIONS  
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## ABSTRACT

Useful formulations of the macroscopic conservation relations for a three-component mixture composed of ions, electrons, and neutral particles are developed from the basic forms of the equations for the individual species. The development represents primarily an orderly compilation of the work of previous investigators, although the equations developed are somewhat more general than those previously derived. In particular, all inertial and viscous terms are retained, and the temperatures of the individual species are not required to be equal. In addition, treatment of the energy equations is included.

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## SECTION I

### ONE-FLUID FORMULATION

The purpose of this study is to develop, in a systematic and general way, useful formulations of the macroscopic conservation relations (i. e. , mass, momentum, and energy) applicable, in the absence of radiative effects, to a gas mixture composed of ions, electrons, and neutral particles. ~~Hence, the material contained in this report~~ constitutes primarily an orderly compilation of the work of previous investigators, ~~see, for example, Schluter,<sup>1</sup> Cowling,<sup>2</sup> Delcroix,<sup>3</sup> and Lehnert.<sup>4</sup>~~ The present work, however, strives to maintain some degree of generality so that the various terms neglected in other treatments are readily apparent. In particular, all inertial and viscous terms are retained, and the temperatures of the species are not required to be equal. In addition, the energy equations, not treated in previous works, are included.

In Section II, the basic equations for each species are written, and arguments are advanced for treating the momentum properties of the mixture either as a single fluid or a two-fluid mixture and the energy properties as a one-, two-, or three-temperature mixture. Forms of the individual species equations convenient for developing the subsequent formulations are presented. In Section III, the one-fluid formulation is developed, and in Section IV, the two-fluid formulation is derived.

## SECTION II

### BASIC MULTICOMPONENT EQUATIONS

#### A. Equations for Each Species

The governing equations for the flow of a multicomponent mixture composed of species  $\alpha$  ( $\alpha = 1, \dots, N$ ) are derived by taking moments of the Boltzmann equation appropriate to each species. In the absence of any external forces except those of electromagnetic origin, the Boltzmann equation for species  $\alpha$  is written as

$$\frac{\partial f_{\alpha}}{\partial t} + \underline{V}_{\alpha} \cdot \underline{\nabla} f + \frac{e_{\alpha}}{m_{\alpha}} (\underline{E} + \underline{V}_{\alpha} \times \underline{B}) \cdot \underline{\nabla}_{\underline{V}} f_{\alpha} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \quad (1)$$

where  $(\partial f / \partial t)_{\text{coll}}$  represents the effects of interactions of members of species  $\alpha$  with all species present, including other members of species  $\alpha$ .

Following the usual procedure, we obtain the transport equations (i. e., continuity, momentum, energy) for each species by multiplying Eq. (1) successively by  $m_{\alpha}$ ,  $m_{\alpha} \underline{V}_{\alpha}$ , and  $1/2 m_{\alpha} \underline{V}_{\alpha}^2 + m_{\alpha} \epsilon_{\alpha}$  and integrating over velocity space, making liberal use of the definition of the mean value of any function of velocity,  $n_{\alpha} \bar{Q} = \int_{-\infty}^{+\infty} f_{\alpha} Q dV_x dV_y dV_z$ . This procedure yields, after suitable rearrangement:

$$\frac{\partial \rho_{\alpha}}{\partial t} + \underline{\nabla} \cdot (\rho_{\alpha} \underline{V}_{\alpha}) = \dot{\omega}_{\alpha} \quad (2)$$

$$\rho_{\alpha} \frac{D^{\alpha} \underline{V}_{\alpha}}{Dt} = - \underline{\nabla} p_{\alpha} + \underline{\nabla} \cdot \underline{T}_{\alpha} + Q_{\alpha} (\underline{E} + \underline{V}_{\alpha} \times \underline{B}) + \sum_{j=1}^N \underline{P}_{\alpha j} - \omega_{\alpha} \underline{V}_{\alpha} \quad (3)$$



$$\rho_a \frac{D^a}{Dt} \left( \hat{u}_a + \epsilon_a \right) = - p_a \left( \nabla \cdot \underline{v}_a \right) - \nabla \cdot \underline{q}_a + \left( \underline{\tau}_a : \underline{\nabla v}_a \right) - \left( \hat{u}_a + \epsilon_a - \frac{v_a^2}{2} \right) \omega_a$$

$$+ \sum_{j=1}^N E_{aj} - \underline{v}_a \cdot \sum_{j=1}^N \underline{P}_{aj} \quad (4a)$$

Other useful forms of the energy equation are obtained in terms of the enthalpy and the total enthalpy

$$\rho_a \frac{D^a}{Dt} \left( \hat{h}_a + \epsilon_a \right) = \underline{v}_a \cdot \underline{\nabla p}_a - \nabla \cdot \underline{q}_a + \left( \underline{\tau}_a : \underline{\nabla v}_a \right) - \left( \hat{h}_a + \epsilon_a - \frac{v_a^2}{2} \right) \omega_a$$

$$+ \sum_{j=1}^N E_{aj} - \underline{v}_a \cdot \sum_{j=1}^N \underline{P}_{aj} + \frac{\partial p_a}{\partial t} \quad (4b)$$

and

$$\rho_a \frac{D^a}{Dt} \left( \hat{h}_a + \frac{v_a^2}{2} + \epsilon_n \right) = - \nabla \cdot \underline{q}_a + \nabla \cdot \left( \underline{v}_a \cdot \underline{\tau}_a \right) + \underline{E} \cdot \underline{Q}_a \underline{v}_a$$

$$- \left( \hat{h}_a + \epsilon_a + \frac{v_a^2}{2} \right) \omega_a + \frac{\partial p_a}{\partial t} + \sum_{j=1}^N E_{aj} \quad (4c)$$

It is to be noted that  $\hat{u}_a$  can also be expressed as  $3kT_a/2m_a$ . The physical significance of the various terms in the above equations is well known and will not be considered here.

To complete the formulation, Maxwell's equations are required:

$$\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t} \quad (5)$$

$$\nabla \times \underline{B} = \mu_o \underline{J} + \mu_o \epsilon_o \frac{\partial \underline{E}}{\partial t} \quad (6)$$

$$\nabla \cdot \underline{E} = \frac{Q}{\epsilon_o} \quad (7)$$

$$\nabla \cdot \underline{B} = 0 \quad (8)$$

Also required are the equations of state for each of the species  $\alpha$ :

$$\hat{u}_\alpha = f(p_\alpha, \rho_\alpha) \quad (9)$$

$$\epsilon_\alpha = f(p_\alpha, \rho_\alpha) \quad (10)$$

It is worth remarking at this point that the basic unknowns of the problem are  $\rho_\alpha$ ,  $\underline{v}_\alpha$ ,  $p_\alpha$ ,  $\underline{E}$ ,  $\underline{B}$ ,  $\hat{u}_\alpha$ ,  $\epsilon_\alpha$ ,  $\underline{T}_\alpha$ ,  $\underline{q}_\alpha$ ,  $\underline{P}_{\alpha j}$ ,  $E_{\alpha j}$ , and  $\omega_\alpha$ ,<sup>†</sup> the entire set of which comprises considerably more unknowns than equations. We shall therefore implicitly assume that solutions of the Boltzmann equation will permit  $\underline{T}_\alpha$ ,  $\underline{q}_\alpha$ ,  $\underline{P}_{\alpha j}$ ,  $E_{\alpha j}$ , and  $\omega_\alpha$  to be expressed in terms of the other unknowns (as is the case for dilute neutral gases). The remaining set of unknowns number  $7N + 6$ , two less than the number of equations available. This is due to the well-known fact that Eq. (8) is merely an initial condition when used with Eq. (5), and the fact that Eqs. (2), (6), and (7) are not independent, since (6) and (7), or (2), can be used to obtain the conservation of charge relation. Hence, either Eq. (5) or (8) and one of Eqs. (2), (6), or (7) may be taken as superfluous equations.

Henceforth, we shall restrict our attention to three-component mixtures composed of ions, electrons, and neutral particles.

#### B. Useful Formulations of the Three-Fluid Equations

The equations for the individual species listed in the previous section (which we shall refer to as the three-fluid, three-temperature formulation) are generally not the most convenient forms for dealing with specific three-component flow problems, due primarily to certain simplifications which

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<sup>†</sup>It is noted that the variables  $Q_\alpha$ ,  $Q$ ,  $\underline{J}$ , and  $n_\alpha$  are defined in terms of the unknowns listed here.

arise from consideration of the coupling between the three species. This coupling, as can be seen from the equations listed, takes three distinct forms: first, momentum coupling due to collisions between particles of different species (the terms  $\underline{P}_{aj}$ ); second, energy coupling due to collisions between particles of different species (the terms  $E_{aj}$ ); and third, electromagnetic coupling due to collective interactions of the charged species [the terms  $Q_a (\underline{E} + \underline{v}_a \times \underline{B})$ ].

If the momentum and electromagnetic coupling between ions and electrons is very strong, then the number density of each species tends to be the same value, and the difference in mean velocities  $\underline{v}_i$  and  $\underline{v}_e$  produced by mechanical effects tends to be small compared to either  $\underline{C}_i$  or  $\underline{C}_e$ . This suggests that it may be useful to describe the momentum properties of the ions and electrons as a single fluid (the plasma) of mass density  $\rho_p = \rho_i + \rho_e$  and mean velocity  $\rho_p \underline{v}_p = \rho_i \underline{v}_i + \rho_e \underline{v}_e$ . The mean velocities of the ions and electrons are then described by the diffusional velocities  $\underline{w}_{ip}$  and  $\underline{w}_{ep}$  where  $\sum_{i,e} \rho_a \underline{w}_{ap} = 0$ .

Similarly, if the momentum coupling between ions and neutrals and/or electrons and neutrals is very strong, the differences in mean velocities  $\underline{v}_i - \underline{v}_n$  and  $\underline{v}_e - \underline{v}_n$  is small compared to  $\underline{C}_i$ ,  $\underline{C}_e$ , or  $\underline{C}_n$ . This suggests that it may be useful to describe the momentum properties of ions, electrons, and neutrals as a single fluid of mass density  $\rho = \rho_i + \rho_e + \rho_n$  and mean velocity  $\rho \underline{v} = \rho_i \underline{v}_i + \rho_e \underline{v}_e + \rho_n \underline{v}_n$ . The mean velocities of the individual species are then described by the diffusional velocities  $\underline{w}_i$ ,  $\underline{w}_e$ , and  $\underline{w}_n$  where  $\sum_{i,e,n} \rho_a \underline{w}_a = 0$ . For this description it is desired to recast the individual momentum equations into the form of one momentum equation for the total mixture, and two additional diffusion equations for determining the diffusional velocities. This is the standard fluid mechanical treatment of multicomponent mixtures (see Hirschfelder, Curtis, and Bird<sup>5</sup>).

If, however, the momentum coupling between the charged particles and neutrals is not strong, it may be desirable to describe the momentum properties of the mixture, composed of (a) a plasma of mass density  $\rho_p$  and mean

velocity  $\underline{v}_p$ , and (b) a neutral gas of mass density  $\rho_n$  and mean velocity  $\underline{v}_n$ . The mean velocities of ions and electrons are described by the diffusional velocities  $\underline{w}_{ip}$  and  $\underline{w}_{ep}$ . For this description it is desired to recast the individual momentum equations for ions and electrons into one momentum equation for the plasma, and one additional diffusion relation (a generalized Ohm's law) for determining the diffusional velocities.

Similar considerations may be applied to the energy coupling, although no exact reformulations appear useful. However, if the coupling between species is sufficiently strong, the assumption  $T_i = T_e = T_n \equiv T$  may be permissible, and the energy properties of the mixture may be described by a single temperature. It is then desirable to combine the individual energy equations into one equation by eliminating the coupling terms. Likewise, if the coupling between ions and neutrals is sufficiently strong to permit the assumption  $T_i = T_n \equiv T_g$  while the electrons are weakly coupled, then the energy properties of the mixture may be described by two temperatures,  $T_g$  and  $T_e$ . In this case it is desirable to combine the energy equations for ions and neutrals by eliminating the mutual coupling term.

There exist, of course, other extremes in both momentum and energy coupling between species which, in theory, would suggest different formulations. These are not considered important here, as they are not physically probable.

To summarize, the momentum and energy properties of a three-component mixture may usefully be described as indicated below.

	<u>Momentum</u>	<u>Energy</u>
Three-fluid:	$\rho_i, \rho_e, \rho_n, \underline{v}_i, \underline{v}_n, \underline{v}_e$	Three-temperature: $T_i, T_e, T_n$
Two-fluid:	$\rho_p, \underline{v}_p, \underline{w}_e, \rho_n, \underline{v}_n, Q$	Two-temperature: $T_e, T_g$
One-fluid:	$\rho, \underline{v}, \underline{w}_e, \underline{w}_n, Q, f$	One-temperature: $T$

### C. Useful Forms for the Individual Component Equations

As an aid in recasting the equations in the forms suitable for the descriptions of the previous section, the individual component continuity, momentum,

and energy equations will be revised in the following ways: first, equations for the ions and electrons will be written in terms of  $\underline{v}_p$  and the diffusional velocities  $\underline{w}_{ip}$  and  $\underline{w}_{ep}$ , and, second, equations for all species will be written in terms of  $\underline{v}$  and the diffusional velocities  $\underline{w}_i$ ,  $\underline{w}_e$ , and  $\underline{w}_n$ .

In terms of the first variables, the equations become (where  $a = i$  or  $e$ )

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \underline{w}_{ap}) + \nabla \cdot (\rho_a \underline{v}_p) = \omega_a \quad (11)$$

$$\begin{aligned} \rho_a \frac{D^p \underline{v}_p}{Dt} + \frac{D^p}{Dt} (\rho_a \underline{w}_{ap}) + \rho_a \underline{w}_{ap} (\nabla \cdot \underline{v}_p) + \rho_a (\underline{w}_{ap} \cdot \nabla) \underline{v}_p = - \nabla p_{ap} \\ + \nabla \cdot \underline{\tau}_{ap} + Q_a (\underline{E} + \underline{v}_p \times \underline{B}) + Q_a \underline{w}_{ap} \times \underline{B} + \sum_j \underline{P}_{aj} - \omega_a \underline{v}_p \end{aligned} \quad (12)$$

$$\begin{aligned} \rho_p \frac{D^p}{Dt} \left[ \frac{\rho_a}{\rho_p} (\hat{u}_{ap} + \epsilon_a) \right] + \rho_a \underline{w}_{ap} \cdot \frac{D^p \underline{v}_p}{Dt} = - \nabla \cdot \underline{q}_{ap} - p_{ap} (\nabla \cdot \underline{v}_p) \\ + (\underline{\tau}_{ap} : \underline{\nabla} \underline{v}_p) + Q_a \underline{w}_{ap} \cdot (\underline{E} + \underline{v}_p \times \underline{B}) - \underline{v}_p \cdot \sum_j \underline{P}_{aj} \\ + \sum_j E_{aj} + \omega_a \frac{v_p^2}{2} - \frac{\rho_a}{\rho_p} (\omega_i + \omega_e) (\hat{u}_{ap} + \epsilon_a) \end{aligned} \quad (13)$$

Additional forms of the energy equation are

$$\begin{aligned} \rho_p \frac{D^p}{Dt} \left[ \frac{\rho_a}{\rho_p} (\hat{h}_{ap} + \epsilon_a) \right] + \rho_a \underline{w}_{ap} \cdot \frac{D^p \underline{v}_p}{Dt} = - \nabla \cdot \underline{q}_{ap} + \underline{v}_p \cdot \nabla p_{ap} \\ + (\underline{\tau}_{ap} : \underline{\nabla} \underline{v}_p) + Q_a \underline{w}_{ap} \cdot (\underline{E} + \underline{v}_p \times \underline{B}) - \underline{v}_p \cdot \sum_j \underline{P}_{aj} \\ + \sum_j E_{aj} + \frac{\partial p_a}{\partial t} + \omega_a \frac{v_p^2}{2} - \frac{\rho_a}{\rho_p} (\omega_i + \omega_e) \left( \hat{u}_{ap} + \epsilon_a + \frac{p_{ap}}{\rho_a} \right) \end{aligned} \quad (14)$$

and

$$\begin{aligned}
& \rho_p \frac{D^p}{Dt} \left[ \frac{\rho_a}{\rho_p} \left( \hat{h}_{ap} + \epsilon_a + \frac{v_p^2}{2} \right) \right] + \rho_p \frac{D^p}{Dt} \left[ \frac{\rho_a}{\rho_p} (\underline{w}_{ap} \cdot \underline{v}_p) \right] \\
& + \nabla \cdot \left( \frac{1}{2} \rho_a v_p^2 \underline{w}_a \right) = - \nabla \cdot \underline{q}_{ap} + \nabla \cdot (\underline{v}_p \cdot \underline{\tau}_{ap}) \\
& + \underline{E} \cdot Q_a \underline{v}_p + \underline{E} \cdot Q_a \underline{w}_{ap} + \sum_j E_{aj} + \frac{\partial p_{ap}}{\partial t} \\
& - \frac{\rho_a}{\rho} (\omega_i + \omega_e) \left[ (\underline{w}_{ap} \cdot \underline{v}_p) + \left( \frac{v_p^2}{2} + \hat{h}_{ap} + \epsilon_a \right) \right] \quad (15)
\end{aligned}$$

It is to be noted that in the above equations the quantities  $p_{ap}$ ,  $\hat{u}_{ap}$ ,  $\hat{h}_{ap}$ ,  $\underline{\tau}_{ap}$ , and  $\underline{q}_{ap}$  are defined in terms of the appropriate mean values of the difference in particle velocity and mean plasma velocity ( $\underline{C}_{ap}$ ), instead of the difference in particle velocity and mean species velocity ( $\underline{C}_a$ ).

In terms of the second variables, the equations become (where  $a = i, e, \text{ or } n$ )

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \underline{w}_a) + \nabla \cdot (\rho_a \underline{v}) = \omega_a \quad (16)$$

$$\begin{aligned}
& \rho_a \frac{D\underline{v}}{Dt} + \frac{D}{Dt} (\rho_a \underline{w}_a) + \rho_a \underline{w}_a (\nabla \cdot \underline{v}) + \rho_a (\underline{w}_a \cdot \nabla) \underline{v} = \\
& - \nabla p_{at} + \nabla \cdot \underline{\tau}_{at} + Q_a (\underline{E} + \underline{v} \times \underline{B}) + Q_a \underline{w}_a \times \underline{B} + \sum_j \underline{P}_{aj} - \omega_a \underline{v} \quad (17)
\end{aligned}$$

$$\begin{aligned} \rho \frac{D}{Dt} \left[ \frac{\rho_a}{\rho} (\hat{u}_{at} + \epsilon_a) \right] + \rho_{a\bar{w}_a} \cdot \frac{D\mathbf{v}}{Dt} = & - \nabla \cdot \mathbf{q}_{at} - p_{at} (\nabla \cdot \mathbf{v}) + (\underline{\tau}_{at} : \nabla \underline{v}) \\ & + Q_{a\bar{w}_a} \cdot (\underline{E} + \mathbf{v} \times \underline{B}) + \sum_j E_{aj} - \mathbf{v} \cdot \sum_j \underline{P}_{aj} + \omega_a \frac{v^2}{2} \end{aligned} \quad (18)$$

Alternative forms of the energy equation are

$$\begin{aligned} \rho \frac{D}{Dt} \left[ \frac{\rho_a}{\rho} (\hat{h}_{at} + \epsilon_a) \right] + \rho_{a\bar{w}_a} \cdot \frac{D\mathbf{v}}{Dt} = & - \nabla \cdot \mathbf{q}_{at} + \mathbf{v} \cdot \nabla p_{at} \\ & + (\underline{\tau}_{at} : \nabla \underline{v}) + Q_{a\bar{w}_a} \cdot (\underline{E} + \mathbf{v} \times \underline{B}) + \sum_j E_{aj} - \mathbf{v} \cdot \sum_j \underline{P}_{aj} \\ & + \omega_a \frac{v^2}{2} + \frac{\partial p_{at}}{\partial t} \end{aligned} \quad (19)$$

and

$$\begin{aligned} \rho \frac{D}{Dt} \left[ \frac{\rho_a}{\rho} \left( \hat{h}_{at} + \frac{v^2}{2} + \epsilon_a \right) \right] + \rho \frac{D}{Dt} \left[ \frac{\rho_a}{\rho} \bar{w}_a \cdot \mathbf{v} \right] + \nabla \cdot \left[ \frac{1}{2} \rho_a v^2 \bar{w}_a \right] = \\ - \nabla \cdot \mathbf{q}_{at} + \nabla \cdot (\mathbf{v} \cdot \underline{\tau}_{at}) + \underline{E} \cdot Q_{a\bar{v}} + \underline{E} \cdot Q_{a\bar{w}_a} \\ + \sum_j E_{aj} + \frac{\partial p_{at}}{\partial t} \end{aligned} \quad (20)$$

Again it is to be noted that the quantities  $p_{at}$ ,  $\hat{u}_{at}$ ,  $\hat{h}_{at}$ ,  $\underline{\tau}_{at}$ , and  $\mathbf{q}_{at}$  are defined in terms of the appropriate mean values of the difference in particle velocity and mean mixture velocity ( $\underline{C}_{at}$ ).

### SECTION III

#### ONE-FLUID FORMULATION

##### A. Basic Fluid Mechanical Formulation

Considering the mixture as a single fluid with momentum properties described by  $\rho$  and  $\underline{v}$ , we obtain the equation of continuity by adding the three Eqs. (16) and making use of the facts that  $\sum \rho_a \underline{w}_a = 0$  and  $\sum \omega_a = 0$ :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad (21)$$

The conservation of the individual species may be expressed directly by (16) where, it is to be remembered, only one such equation is required in combination with Eq. (21) and Maxwell's equations.

The momentum equation for the mixture is obtained by adding the Eqs.(17):

$$\rho \frac{D\underline{v}}{Dt} = - \nabla P_t + \nabla \cdot \underline{\tau}_t + Q(\underline{E} + \underline{v} \times \underline{B}) + \underline{j} \times \underline{B} \quad (22)$$

A form for the diffusional relations useful for comparison with standard fluid mechanical developments is obtained by eliminating the  $D\underline{v}/Dt$  term from Eqs. (17) and (22). (Only two of these relations are independent.)

$$\begin{aligned} \sum_j \underline{P}_{aj} = & - \frac{\rho_a}{\rho} \nabla P_t + \nabla P_{at} + \left( \frac{\rho_a}{\rho} Q - Q_a \right) (\underline{E} + \underline{v} \times \underline{B}) \\ & + \left( \frac{\rho_a}{\rho} \underline{j} - Q_a \underline{w}_a \right) \times \underline{B} + \omega_a \underline{v} + \frac{\rho_a}{\rho} (\nabla \cdot \underline{\tau}_t) - (\nabla \cdot \underline{\tau}_{at}) \\ & + \frac{D}{Dt} (\rho_a \underline{w}_a) + \rho_a \underline{w}_a (\nabla \cdot \underline{v}) + \rho_a (\underline{w}_a \cdot \nabla) \underline{v} \end{aligned} \quad (23)$$



These equations merely represent the conservation of the difference in momentum of a single species and the mixture; the left-hand side is the friction force acting on a single species due to interaction with the other species, and the right-hand side represents the summation of effects which require this interaction for dynamical equilibrium. In addition, these equations are of the form obtained by Hirschfelder, Curtiss, and Bird<sup>5</sup> by the standard perturbation solution of the Boltzmann equations for the species, in which it is assumed that:

1. All species are of equal temperature;
2. The velocity distribution function of each species is perturbed slightly from the Maxwellian one corresponding to the mean properties of the mixture;
3. The forces on the particles arising due to  $\underline{E}$  and  $\underline{B}$  do not directly affect the distribution function;
4. There is no chemical reaction ( $\omega_a = 0$ ).

The precise equations obtained by Hirschfelder, Curtiss, and Bird,<sup>5</sup> when written in the form of Eqs.(23), are:

$$\begin{aligned} \sum_j \underline{P}_{aj} = & - \frac{\rho_a}{\rho} \nabla p_t + \nabla p_{at} + \left( \frac{\rho_a}{\rho} Q - Q_a \right) (\underline{E} + \underline{v} \times \underline{B}) \\ & + \left( \frac{\rho_a}{\rho} \underline{j} - Q_a \underline{w}_a \times \underline{B} \right) \end{aligned} \quad (24)$$

where

$$\underline{P}_{aj} \equiv \frac{n_a n_j}{n^2 D_{aj}(1)} (\underline{w}_j - \underline{w}_a) + \frac{n_a n_j}{n^2 D_{aj}(1)} \left( \frac{D_j^T}{\rho_j} - \frac{D_a^T}{\rho_a} \right) \frac{\nabla T}{T} \quad (25)$$

The usefulness of comparing Eqs. (24) and (25) with Eq. (23) is that it makes evident the effect of the approximations made in obtaining (24) and (25) [the last five terms on the right-hand side of (23) are neglected], and it further yields the form of the friction terms  $\underline{P}_{aj}$  (subject, of course, to the previous assumptions). It is pointed out that the terms neglected in Eq. (23) are a direct consequence of the assumption that the velocity distribution function of each species suffers only small departures from the Maxwellian one corresponding to the mean properties of the mixture. If the coupling between species is strong, we may expect the mean properties of each species to differ little from the mean mixture properties; this, of course, represents the situation in which the one-fluid model is convenient.

For the one-temperature model, the energy equations are obtained by adding the Eqs. (18), (19), or (20), respectively:

$$\rho \frac{D}{Dt} \left[ \sum \frac{\rho_a}{\rho} \left( \hat{u}_{at} + \epsilon_a \right) \right] = - \nabla \cdot \underline{q}_t - p_t (\nabla \cdot \underline{v}) + (\underline{\tau}_t : \underline{\nabla v}) + \underline{j} \cdot (\underline{E} + \underline{v} \times \underline{B}) \quad (26)$$

or

$$\rho \frac{D}{Dt} \left[ \sum \frac{\rho_a}{\rho} \left( \hat{h}_{at} + \epsilon_a \right) \right] = - \nabla \cdot \underline{q}_t + \underline{v} \cdot \underline{\nabla} p_t + (\underline{\tau}_t : \underline{\nabla v}) + \underline{j} \cdot (\underline{E} + \underline{v} \times \underline{B}) + \frac{\partial p_t}{\partial t} \quad (27)$$

or

$$\rho \frac{D}{Dt} \left[ \sum \frac{\rho_a}{\rho} \left( \hat{h}_{at} + \epsilon_a + \frac{v^2}{2} \right) \right] = - \nabla \cdot \underline{q}_{at} + \nabla \cdot (\underline{v} \cdot \underline{\tau}_t) + \underline{E} \cdot \underline{Qv} + \underline{E} \cdot \underline{j} + \frac{\partial p_t}{\partial t} \quad (28)$$

Similarly, the appropriate energy equations for the two-temperature model are obtained by adding Eqs. (18), (19), or (20) for ions and neutrals, respectively:

$$\begin{aligned}
\rho \frac{D}{Dt} \left[ \sum_{i,n} \frac{\rho_a}{\rho} \left( \hat{u}_{at} + \epsilon_a \right) \right] - \rho_e \underline{w}_e \cdot \frac{D\underline{v}}{Dt} = & - \nabla \cdot (\underline{q}_{it} + \underline{q}_{nt}) \\
& - (p_{it} + p_{nt}) (\nabla \cdot \underline{v}) + [(\underline{\tau}_{it} + \underline{\tau}_{nt}) : \underline{\nabla v}] \\
& + \underline{Q}_i \underline{w}_i \cdot (\underline{E} + \underline{v} \times \underline{B}) + E_{ie} + E_{ne} - \underline{v} \cdot (\underline{P}_{ie} + \underline{P}_{ne}) \\
& + (\omega_i + \omega_n) \frac{v^2}{2}
\end{aligned} \tag{29}$$

or

$$\begin{aligned}
\rho \frac{D}{Dt} \left[ \sum_{i,n} \frac{\rho_a}{\rho} \left( \hat{h}_{at} + \epsilon_a \right) \right] - \rho_e \underline{w}_e \cdot \frac{D\underline{v}}{Dt} = & - \nabla \cdot (\underline{q}_{it} + \underline{q}_{nt}) + \underline{v} \cdot \nabla (p_{it} + p_{nt}) \\
& + [(\underline{\tau}_{it} + \underline{\tau}_{nt}) : \underline{\nabla v}] + \underline{Q}_i \underline{w}_i \cdot (\underline{E} + \underline{v} \times \underline{B}) + E_{ie} + E_{ne} \\
& - \underline{v} \cdot (\underline{P}_{ie} + \underline{P}_{ne}) + (\omega_i + \omega_n) \frac{v^2}{2} + \frac{\partial}{\partial t} (p_{it} + p_{nt})
\end{aligned} \tag{30}$$

or

$$\begin{aligned}
\rho \frac{D}{Dt} \left[ \sum_{i,n} \frac{\rho_a}{\rho} \left( \hat{h}_{at} + \epsilon_a + \frac{v^2}{2} \right) \right] - \rho \frac{D}{Dt} \left( \frac{\rho_e}{\rho} \underline{w}_e \cdot \underline{v} \right) - \nabla \cdot \left( \frac{1}{2} \rho_e \underline{w}_e v^2 \right) = \\
- \nabla \cdot (\underline{q}_{it} + \underline{q}_{nt}) + \nabla \cdot \left[ \underline{v} \cdot (\underline{\tau}_{it} + \underline{\tau}_{nt}) \right] \\
+ \underline{E} \cdot \underline{Q}_a \underline{v} + \underline{E} \cdot \underline{Q}_i \underline{w}_i + E_{ie} + E_{ne} + \frac{\partial(p_{it} + p_{nt})}{\partial t}
\end{aligned} \tag{31}$$

The energy equation for the electrons is obtained directly from Eqs. (18), (19), or (20).

For the three-temperature model, the appropriate energy equations are obtained directly from Eqs. (18), (19), or (20).

#### B. Form of the Friction Forces, $\underline{P}_{aj}$

It is not profitable to proceed with the one-fluid model without introducing the form of the momentum coupling terms  $\underline{P}_{aj}$ . The form adopted here, for reasons to be discussed, is as follows:

$$\underline{P}_{ie} = - \underline{P}_{ei} = \underline{\eta}_{ie} \cdot (\underline{w}_e - \underline{w}_i) + \bar{\underline{P}}_{ie} \tag{32}$$

$$\underline{P}_{in} = - \underline{P}_{ni} = \underline{\eta}_{in} (\underline{w}_n - \underline{w}_i) + \bar{\underline{P}}_{in} \tag{33}$$

$$\underline{P}_{ne} = - \underline{P}_{en} = \underline{\eta}_{en} \cdot (\underline{w}_e - \underline{w}_n) + \bar{\underline{P}}_{ne} \tag{34}$$

where the tensors  $\underline{\eta}_{ie}$  and  $\underline{\eta}_{en}$  are of the form

$$\underline{\eta}_{aj} = \eta_{\perp aj} \underline{\underline{I}} - (\eta_{\parallel aj} - \eta_{\perp aj}) \underline{nn} \tag{35}$$

where  $\underline{I}$  is the unit tensor and  $\underline{n}$  is the unit vector in the direction of the magnetic field.

The form of these relations suggested by comparison with the standard fluid mechanical treatment is, from Eq. (25),  $P_{aj} = \eta_{aj}(\underline{w}_j - \underline{w}_a) + \eta_{aj}^T \nabla T$ . It is recalled, however, that this form includes the assumption that electric and magnetic fields do not directly affect the velocity distribution function; this has been shown by other investigators (e. g. , Chapman and Cowling,<sup>6</sup> Marshall,<sup>7</sup> and Braginski<sup>8</sup>) to be imprecise. In particular, Braginski obtains, for a mixture of ions and electrons, the following form:

$$\begin{aligned} \underline{P}_{aj} = & \underline{\eta}_{aj} \cdot (\underline{w}_j - \underline{w}_a) + \eta'_{aj} \left[ (\underline{w}_j - \underline{w}_a) \times \frac{\underline{B}}{|B|} \right] \\ & + \underline{\eta}_{aj}^T \cdot \underline{\nabla T} + \eta_{aj}^T \left( \underline{\nabla T} \times \frac{\underline{B}}{|B|} \right) \end{aligned} \quad (36)$$

where it has been assumed that the velocity distribution function is unaffected by electric fields. The form of the tensors  $\underline{\eta}$  is

$$\underline{\eta} = \eta_{\perp} \underline{I} - (\eta_{\parallel} - \eta_{\perp}) \underline{nn} \quad (37)$$

Clearly, if the electric fields were properly accounted for, a term proportional to  $\underline{E}$  is expected in Eq. (36).

Examination of Braginski's results reveals that over all ranges of magnetic field strength,  $\eta_{\perp}$  in (37) varies from  $\eta_{\parallel}$  to  $2\eta_{\parallel}$ ;  $\eta'_{aj}$  is always less than 20 percent of the value of  $\eta_{\parallel}$ . Furthermore, for moderate magnetic field strengths  $[(\omega \tau)_{ions} < 1]$ , the ion velocity distribution function is only slightly affected by the magnetic field. These considerations suggest that the forms of  $\underline{P}_{aj}$  in Eqs. (32) through (34) give proper emphasis to the most important effects.

C. One-Fluid Formulation in Terms of  $\rho$ ,  $f$ ,  $Q$ ,  $\underline{v}$ ,  $\underline{j}$ , and  $\underline{v}_A$

1. Relationships Between Variables

It is found most convenient to cast the one-fluid formulation in terms of the variables  $\rho$ ,  $f$ ,  $Q$ ,  $\underline{v}$ ,  $\underline{j}$ , and  $\underline{v}_A$ . We list here the other important variables in terms of these:

$$\rho_i = a_2^* (1 - f)\rho \quad (38a)$$

$$\rho_n = a_3^* f\rho \quad (38b)$$

$$\rho_e = - a_2^* a_6^{**} \frac{m_e e_i}{m_i e_e} (1 - f)\rho \quad (38c)$$

$$Q_i = a_2^* \frac{e_i}{m_i} (1 - f)\rho \quad (38d)$$

$$Q_e = Q - a_2^* \frac{e_i}{m_i} (1 - f)\rho \quad (38e)$$

$$\underline{w}_i = - \frac{a_5}{a_2^*} \frac{m_e}{e_e} \frac{\underline{j}}{(1 - f)\rho} + \frac{a_5 a_4^*}{a_2^*} \frac{(1 - f a_2^*)}{(1 - f)} \underline{v}_A \quad (38f)$$

$$\underline{w}_n = - \frac{a_4^* (1 - f a_2^*)}{a_3^* f} \underline{v}_A \quad (38g)$$

$$\underline{w}_e = - \frac{a_5 \underline{j}}{a_2^* a_6^{**} \frac{e_i}{m_i} (1 - f)\rho} + \frac{a_5 a_4^* (1 - f a_2^*)}{a_6^{**} (1 - f) a_2^*} \underline{v}_A \quad (38h)$$

where

$$a_1^* \equiv 1 - Q \frac{m_e}{\rho e_e} \quad (39a)$$

$$a_2^* \equiv \frac{a_1^*}{1 + \frac{m_e}{m_i} f - \frac{m_e e_i}{m_i e_e} (1 - f)} \quad (39b)$$

$$a_3^* \equiv \frac{m_n}{m_i} a_2^* \quad (39c)$$

$$a_4^* \equiv 1 + \frac{a_2^* - a_1^*}{1 - a_2^* f} - \left( \frac{a_5 - 1}{a_5} \right) a_2^* \frac{1 - f}{(1 - f a_2^*)} \quad (39d)$$

$$a_5 \equiv \left( 1 - \frac{e_i m_e}{e_e m_i} \right)^{-1} \quad (39e)$$

$$a_6^{**} \equiv 1 - \frac{m_i Q}{a_2^* e_i \rho (1 - f)} \quad (39f)$$

Furthermore, we may write

$$\underline{w}_i - \underline{w}_e = \frac{A_{ie}^*}{a_6^{**}} \frac{m_i}{e_i (1 - f) \rho} \underline{j} + D_{ie}^* \left( 1 - \frac{1}{a_6^{**}} \right) \underline{v}_A \quad (40a)$$

$$\underline{w}_n - \underline{w}_e = \frac{a_5}{a_2^* a_6^{**}} \frac{m_i}{e_i (1 - f) \rho} \underline{j} - \frac{D_{ne}^{**}}{f} \underline{v}_A \quad (40b)$$

$$\underline{w}_n = \underline{w}_i + \frac{a_5^*}{a_2^*} \frac{m_e}{e_e (1-f)\rho} \underline{j} = \frac{D_{in}^*}{f} \underline{v}_A \quad (40c)$$

where

$$A_{ie}^* = \frac{a_5^*}{a_2^*} \left( 1 - \frac{a_6^{**} m_e e_i}{m_i e_e} \right) \quad (41a)$$

$$D_{ie}^* = \frac{a_5^* a_4^*}{a_2^*} \frac{(1 - f a_2^*)}{(1 - f)} \quad (41b)$$

$$D_{ne}^{**} = \frac{a_4^*}{a_3^*} - f \frac{a_4^* a_2^*}{a_3^*} + f \frac{a_5^* a_4^*}{a_2^* a_6^{**}} \frac{(1 - f a_2^*)}{(1 - f)} \quad (41c)$$

$$D_{in}^* = \frac{a_4^*}{a_3^*} - f \frac{a_4^* a_2^*}{a_3^*} + f \frac{a_5^* a_4^*}{a_2^*} \frac{(1 - f a_2^*)}{(1 - f)} \quad (41d)$$

For the sake of brevity, we shall now make the standard assumption of charge neutrality (see Lehnert<sup>4</sup>) and further assume that  $m_e/m_i \ll 1$  and  $|m_e e_i/m_i e_e| \ll 1$ . In the absence of the assumption of charge neutrality, there appears to be no advantage in the one-fluid formulation. With these assumptions, the coefficients  $\alpha$ ,  $A$ , and  $D$  in the above equations become unity.<sup>†</sup> We shall neglect the unspecified terms in the friction forces between species ( $\bar{P}_{aj}$ ).

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<sup>†</sup>In order to obtain the correct limits, it is necessary to perform the subsequent algebra by retaining the coefficients  $\alpha$ ,  $A$ , and  $D$ , and then invoking the previous assumptions.



## 2. Generalized Ohm's Law and Ambipolar Diffusion Equation

In terms of the new variables, it is convenient first to extract an Ohm's law for  $\underline{j}$  and a diffusion equation for  $\underline{v}_A$  from the three dependent relations (23), where Eqs. (32) through (34) with (40a-c) are used to eliminate  $\underline{P}_{aj}$ . To obtain a diffusion equation for  $\underline{v}_A$ , Eq. (23) for neutrals is written in terms of the new variables, with the result that

$$\begin{aligned}
 \frac{(\eta_{en||,\perp} + \eta_{in})}{f} \underline{v}_{A||,\perp} = & - f \frac{\nabla p_{t||,\perp}}{\rho} + \frac{\nabla p_{nt||,\perp}}{\rho} + f(\underline{j} \times \underline{B})_{||,\perp} \\
 & + \left( \eta_{en||,\perp} + \frac{m_e e_i}{m_i e_e} \eta_{in} \right) \frac{m_i}{e_i(1-f)\rho} \underline{j}_{||,\perp} \\
 & + \omega_n \underline{v}_{||,\perp} + f(\nabla \cdot \underline{T}_t)_{||,\perp} - (\nabla \cdot \underline{T}_{nt})_{||,\perp} \\
 & - \left\{ \frac{D}{Dt} \left[ \rho(1-f)\underline{v}_A \right] + \rho(1-f) \right. \\
 & \left. \times \left[ \underline{v}_A(\nabla \cdot \underline{v}) + (\underline{v}_A \cdot \nabla)\underline{v} \right] \right\} \quad (42)
 \end{aligned}$$

To obtain a generalized Ohm's law, Eq. (23) for ions is subtracted from the similar one for electrons, and Eq. (42) is employed to eliminate  $\underline{v}_A$ . The result is

$$\begin{aligned}
\frac{j_{\parallel, \perp}}{\sigma_{\parallel, \perp}} = & (\underline{E} + \underline{v} \times \underline{B})_{\parallel, \perp} - \frac{m_i}{e_i \rho(1-f)} (1 - 2 f \gamma_{\parallel, \perp}) (\underline{j} \times \underline{B})_{\parallel, \perp} \\
& + (\underline{E}_p)_{\parallel, \perp} + (\underline{E}_\tau)_{\parallel, \perp} + \left[ (1 - \gamma_{\parallel, \perp}) \omega_e - \gamma_{\parallel, \perp} \omega_i \right] \frac{m_i}{e_i \rho(1-f)} \underline{v}_{\parallel, \perp} \\
& + \frac{m_e m_i}{e_e e_i \rho(1-f)} \left[ \frac{D \underline{j}}{Dt} + \underline{j}(\nabla \cdot \underline{v}) + (\underline{j} \cdot \nabla) \underline{v} \right]_{\parallel, \perp} \\
& - \gamma_{\parallel, \perp} \left\{ \frac{D}{Dt} \left[ \rho(1-f) \underline{v}_A \right] + \rho(1-f) \left[ \underline{v}_A (\nabla \cdot \underline{v}) + (\underline{v}_A \cdot \nabla) \underline{v} \right] \right\}_{\parallel, \perp} \\
& + \frac{f^2}{\eta_{en\parallel, \perp} + \eta_{in}} \left\{ \left[ \underline{F}_p + \underline{F}_\tau + (\underline{j} \times \underline{B}) + \frac{\omega_n}{f} \underline{v} - \frac{1}{f} \left\{ \frac{D}{Dt} \left[ \rho(1-f) \underline{v}_A \right] \right. \right. \right. \\
& \left. \left. \left. + \rho(1-f) \left[ \underline{v}_A (\nabla \cdot \underline{v}) + (\underline{v}_A \cdot \nabla) \underline{v} \right] \right\} \right] \times \underline{B} \right\}_{\parallel, \perp}, \quad (43)
\end{aligned}$$

where

$$\gamma_{\parallel, \perp} = \frac{\eta_{en\parallel, \perp}}{\eta_{en\parallel, \perp} + \eta_{in}} \quad (44)$$

and

$$\begin{aligned}
(\sigma_{\parallel, \perp})^{-1} = & \left[ \frac{m_i}{e_i(1-f)\rho} \right]^2 \\
& \times \left[ (\eta_{ie})_{\parallel, \perp} + (\eta_{en})_{\parallel, \perp} (1 - \gamma_{\parallel, \perp}) - \gamma_{\parallel, \perp} \eta_{in} \left( \frac{m_e e_i}{m_i e_e} \right) \right] \quad (45a)
\end{aligned}$$

$$(\underline{E}_p)_{\parallel, \perp} = \frac{m_i}{e_i \rho(1-f)} \left[ \gamma_{\parallel, \perp} (1-f) (\nabla p_t)_{\parallel, \perp} - \gamma_{\parallel, \perp} (\nabla p_{it})_{\parallel, \perp} + \frac{e_i m_e}{e m_i} (\nabla p_{it}) + (1 - \gamma_{\parallel, \perp}) (\nabla p_{et})_{\parallel, \perp} \right] \quad (45b)$$

$$(\underline{E}_\tau)_{\parallel, \perp} = - \frac{m_i}{e_i \rho(1-f)} \left[ \gamma_{\parallel, \perp} (1-f) (\nabla \cdot \underline{\tau}_t)_{\parallel, \perp} - \gamma_{\parallel, \perp} (\nabla \cdot \underline{\tau}_{it})_{\parallel, \perp} + \frac{e_i m_e}{e m_i} (\nabla \cdot \underline{\tau}_{it}) + (1 - \gamma_{\parallel, \perp}) (\nabla \cdot \underline{\tau}_{et})_{\parallel, \perp} \right] \quad (45c)$$

$$\underline{F}_p = \frac{1-f}{f} \nabla p_t - \frac{1}{f} \nabla p_{it} - \frac{1}{f} \nabla p_{et} \quad (45d)$$

$$\underline{F}_\tau = - \left( \frac{1-f}{f} \right) (\nabla \cdot \underline{\tau}_t) + \frac{1}{f} (\nabla \cdot \underline{\tau}_{it}) + \frac{1}{f} (\nabla \cdot \underline{\tau}_{et}) \quad (45e)$$

Schluter<sup>1</sup> has derived equations similar to Eq. (43), in which  $\eta_{en}$  and  $\eta_{ie}$  are scalars and the viscous stresses are neglected. Cowling<sup>2</sup> has employed Schluter's assumptions in deriving similar equations, assuming in addition that the temperatures of all components are equal and that the inertial terms involving  $\underline{j}$  and  $\underline{v}_A$  are neglected. Briefly, the terms in this equation represent current-producing mechanisms due to: (a) electric field, (b) Hall effect, (c) different motions of ions and electrons caused by pressure gradients, (d) different motions of ions and electrons caused by viscous stresses, (e) generation of ions and electrons of unequal velocities, (f) different inertial forces acting on ions and electrons, (g) different inertial forces acting in the ion-electron mixture and the neutral particles and (h) relative motion between ions and neutral particles. The form of this equation for a fully ionized gas is readily obtained by setting

$$f = \frac{v_A}{\omega_i} = \frac{\omega_e}{\omega_i} = 0.$$

In Eq. (42), the various terms represent motion of the plasma relative to the mixture due to: (a) action of pressure gradients (this includes both mass diffusion and pressure diffusion), (b) the Lorentz force acting on the plasma, (c) currents flowing in the plasma, (d) generation of neutral species (i. e., recombination of ions and electrons), (e) action of viscous stresses, and (f) inertial effects. Schluter has also derived a similar equation, and has pointed out that, if the last five effects are absent, the temperatures of all components are equal, and the gas is slightly ionized ( $f \ll 1$ ), then Eq. (42) reduces to the usual ambipolar diffusion equation:

$$\underline{v}_{A \parallel, \perp} = - \left( \frac{2kT}{\eta_{en \parallel, \perp} + \eta_{in}} \right) \underline{\nabla} n_{i \parallel, \perp}$$

### 3. Continuity and Momentum Equations

Incorporation of the previous assumptions leaves the continuity equation unchanged; hence, from Eq. (21)

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \quad (46)$$

Recalling that only one species-conservation relation is necessary, we obtain from Eq. (16), for the neutral species:

$$\frac{\partial}{\partial t} (f\rho) + \underline{\nabla} \cdot [\rho(1 - f)\underline{v}_A] = \omega_n - (\underline{\nabla} \cdot f\rho \underline{v}) \quad (47)$$

The momentum equation is obtained from Eq. (22) with  $Q = 0$

$$\rho \frac{D\underline{v}}{Dt} = - \underline{\nabla} p_t + \underline{\nabla} \cdot \underline{\tau}_t + (\underline{j} \times \underline{B}) \quad (48)$$

#### 4. Energy Equations

The energy equations for the three-temperature model are obtained directly from Eqs. (18), (19), or (20) using the relations (38) through (41). The form corresponding to Eq. (20) becomes, for ions:

$$\begin{aligned}
 & \rho \frac{D}{Dt} \left[ (1 - f) \left( \hat{h}_{it} + \frac{v^2}{2} + \epsilon_i \right) \right] + \rho \frac{D}{Dt} \left[ \left\{ (1 - f) \underline{v}_A - \frac{m_e}{e_e \rho} \underline{j} \right\} \cdot \underline{v} \right] \\
 & + \nabla \cdot \left\{ \frac{1}{2} \rho v^2 \left[ (1 - f) \underline{v}_A - \frac{m_e}{\rho e_e} \underline{j} \right] \right\} = - \nabla \cdot \underline{q}_{it} + \nabla \cdot (\underline{v} \cdot \underline{\tau}_{it}) \\
 & + \frac{e_i}{m_i} \rho (1 - f) (\underline{E} \cdot \underline{v}) + \frac{e_i}{m_i} \rho \left[ \underline{E} \cdot \left\{ (1 - f) \underline{v}_A - \frac{m_e}{\rho e_e} \underline{j} \right\} \right] \\
 & + \underline{E}_{in} + \underline{E}_{ie} + \frac{\partial p_{it}}{\partial t}
 \end{aligned} \tag{49a}$$

for electrons:

$$\begin{aligned}
 & \rho \frac{D}{Dt} \left[ - \frac{m_e e_i}{m_i e_e} (1 - f) \left( \hat{h}_{et} + \frac{v^2}{2} + \epsilon_e \right) \right] + \rho \frac{D}{Dt} \left\{ - \frac{m_e e_i}{m_i e_e} \left[ (1 - f) \underline{v}_A \right. \right. \\
 & \left. \left. - \frac{m_i}{e_i \rho} \underline{j} \right] \cdot \underline{v} \right\} + \nabla \cdot \left\{ \frac{1}{2} \rho v^2 \left( - \frac{m_e e_i}{m_i e_e} \right) \left[ (1 - f) \underline{v}_A - \frac{m_i}{e_i \rho} \underline{j} \right] \right\} = \\
 & - \nabla \cdot \underline{q}_{et} + \nabla \cdot (\underline{v} \cdot \underline{\tau}_{et}) - \frac{e_i}{m_i} (1 - f) \rho (\underline{E} \cdot \underline{v}) - \frac{e_i}{m_i} \rho \\
 & \times \left[ \underline{E} \cdot \left\{ (1 - f) \underline{v}_A - \frac{m_i}{e_i \rho} \underline{j} \right\} \right] + \underline{E}_{en} - \underline{E}_{ie} + \frac{\partial p_{et}}{\partial t}
 \end{aligned} \tag{49b}$$

and, for neutrals:

$$\begin{aligned} \rho \frac{D}{Dt} \left[ f \left( \hat{h}_{nt} + \frac{v^2}{2} + \epsilon_n \right) \right] - \rho \frac{D}{Dt} \left[ (1-f) \underline{v}_A \cdot \underline{v} \right] - \nabla \cdot \left[ \frac{1}{2} \rho v^2 (1-f) \underline{v}_A \right] = \\ - \nabla \cdot \underline{q}_{nt} + \nabla \cdot (\underline{v} \cdot \underline{\tau}_{nt}) - E_{en} - E_{in} + \frac{\partial p_{nt}}{\partial t} \end{aligned} \quad (49c)$$

Relations corresponding to Eqs. (18) and (19) can similarly be obtained.

The energy equations for the two-temperature model are obtained from Eqs. (29), (30), or (31). For the mixture of ions and neutrals, the form corresponding to Eq. (31) becomes

$$\begin{aligned} \rho \frac{D}{Dt} \left[ (1-f) \left( \hat{h}_{it} + \epsilon_i \right) + f \left( \hat{h}_{it} + \epsilon_n \right) + \frac{v^2}{2} \right] \\ - \rho \frac{D}{Dt} \left\{ - \frac{m_e e_i}{m_i e_e} \left[ (1-f) \underline{v}_A - \frac{m_i}{e_i \rho} \underline{j} \right] \cdot \underline{v} \right\} - \nabla \cdot \left[ \frac{1}{2} \rho v^2 \left( - \frac{m_e e_i}{m_i e_e} \right) \right. \\ \left. \times \left[ (1-f) \underline{v}_A - \frac{m_i}{e_i \rho} \underline{j} \right] \right] = - \nabla \cdot (\underline{q}_{it} + \underline{q}_{nt}) + \nabla \cdot [\underline{v} \cdot (\underline{\tau}_{it} + \underline{\tau}_{nt})] \\ + \frac{e_i}{m_i} \rho (1-f) (\underline{E} \cdot \underline{v}) + \frac{e_i}{m_i} \rho \left\{ \underline{E} \cdot \left[ (1-f) \underline{v}_A - \frac{m_e}{\rho e_e} \underline{j} \right] \right\} \\ + \underline{E}_{ie} + \underline{E}_{ne} + \frac{\partial (p_{it} + p_{nt})}{\partial t} \end{aligned} \quad (50)$$

The energy equation for electrons for this case is again (49b).

The one-temperature formulation for the energy equation is obtained from Eqs. (26), (27), or (28). The form corresponding to Eq. (28) becomes

$$\rho \frac{D}{Dt} \left[ (1-f) (\hat{h}_{it} + \epsilon_i) - \frac{m_e e_i}{m_i e_e} (1-f) (\hat{h}_{et} + \epsilon_e) + f (\hat{h}_{nt} + \epsilon_n) + \frac{v^2}{2} \right] =$$

$$- \nabla \cdot \underline{q}_t + \nabla \cdot (\underline{v} \cdot \underline{\tau}_t) + \underline{E} \cdot \underline{j} + \frac{\partial p_t}{\partial t} \quad (51)$$

It is pointed out that the relationships of the energy flux vectors  $\underline{q}_{it}$ ,  $\underline{q}_{et}$ , and  $\underline{q}_{nt}$  to the corresponding vectors relative to the individual species  $\underline{q}_i$ ,  $\underline{q}_e$ , and  $\underline{q}_n$  are given by:

$$\underline{q}_{it} = \underline{q}_i - \left[ \underline{v}_A - \frac{m_e}{e_e(1-f)\rho} \underline{j} \right] \cdot \underline{\tau}_{it} + \left[ \rho(1-f)\underline{v}_A - \frac{m_e}{e_e} \underline{j} \right] (\hat{h}_{it} + \epsilon_i)$$

$$\underline{q}_{et} = \underline{q}_e - \left[ \underline{v}_A - \frac{m_i}{e_i(1-f)\rho} \underline{j} \right] \cdot \underline{\tau}_{et}$$

$$+ \left[ \left( - \frac{e_i m_e}{e_e m_i} \right) \rho(1-f)\underline{v}_A + \frac{m_e}{e_e} \underline{j} \right] (\hat{h}_{et} + \epsilon_e)$$

$$\underline{q}_{nt} = \underline{q}_n + \frac{1-f}{f} \underline{v}_A \cdot \underline{\tau}_{nt} - \rho(1-f)\underline{v}_A (\hat{h}_{nt} + \epsilon_n)$$

where the terms represent (a) energy flux due to transport within species, (b) work performed by shear forces due to relative motions of species, and (c) transport of energy due to relative motion of species. Of these, the term

$$\frac{m_e}{e_e} \underline{j} (\hat{h}_e) \approx \frac{5}{2} \frac{kT_e}{e} \underline{j}$$

is the most widely recognized.

## 5. Concluding Remarks on the One-Fluid Formulation

In addition to the continuity, momentum, energy, and diffusion equations presented previously, Maxwell's Eqs. (5) through (7) [Eq. (7) is unnecessary if  $Q = 0$ ] and the equations of state (9) through (10) are required. In the latter connection, it should again be noted that the quantities  $p_t$ ,  $p_{it}$ ,  $p_{et}$ ,  $p_{nt}$ , et cetera, differ from the corresponding ones  $p$ ,  $p_i$ ,  $p_e$ ,  $p_n$ , et cetera, by factors containing the diffusional velocities squared ( $\underline{w}_d^2$ ); hence these factors must be negligible compared to  $\bar{C}_a^2$  if any confidence is to be placed in the equations of state.



## SECTION IV

### TWO-FLUID FORMULATION

#### A. Basic Formulation

The two-fluid model (i. e., plasma and neutral gas) is best formulated in terms of the variables  $\rho_p$ ,  $\rho_n$ ,  $Q$ ,  $\underline{v}_p$ ,  $\underline{v}_n$ , and  $\underline{j}_p$ . For future reference, the relations between the diffusional velocities and plasma current are

$$\underline{w}_{ep} = \frac{1}{\rho_e} \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) \underline{j}_p \quad (52a)$$

and

$$\underline{w}_{ip} = - \frac{1}{\rho_i} \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) \underline{j}_p \quad (52b)$$

and the relations between ion and electron densities and the plasma density and net charge density are

$$\frac{\rho_e}{\rho_p} = \frac{\frac{Q}{\rho_p} - \frac{e_i}{m_i}}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \quad (52c)$$

and

$$\frac{\rho_i}{\rho_p} = \frac{\frac{e_e}{m_e} - \frac{Q}{\rho_p}}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \quad (52d)$$

The continuity equation for the plasma is obtained by adding Eqs. (11) for ions and electrons (noting that  $\sum_{i,e} \rho_a \underline{w}_{ap} = 0$ )

$$\frac{\partial \rho_p}{\partial t} = \nabla \cdot (\rho_p \underline{v}_p) = \omega_i + \omega_e \quad (53)$$

The continuity equation for the neutral gas is obtained directly from Eq. (2)

$$\frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_n \underline{v}_n) = \omega_n \quad (54)$$

Addition of Eqs. (12) for ions and for electrons yields the momentum equation for the plasma

$$\begin{aligned} \rho_p \frac{D^p \underline{v}_p}{Dt} = & - \nabla p_p + \nabla \cdot \underline{\tau}_p + Q(\underline{E} + \underline{v}_p \times \underline{B}) + (\underline{j}_p \times \underline{B}) \\ & + \underline{P}_{in} + \underline{P}_{en} - (\omega_i + \omega_e) \underline{v}_p \end{aligned} \quad (55)$$

The momentum equation for the neutral gas is obtained directly from Eq. (3)

$$\rho_n \frac{D^n \underline{v}_n}{Dt} = - \nabla p_n + (\nabla \cdot \underline{\tau}_n) - \underline{P}_{in} - \underline{P}_{en} - \omega_n \underline{v}_n \quad (56)$$

The appropriate form of a generalized Ohm's law is obtained by multiplying Eq. (12) for ions by  $e_i/m_i$ , multiplying Eq. (12) for electrons by  $e_e/m_e$ , adding the two results, eliminating the term in this result containing  $D^p \underline{v}_p / Dt$  with Eq. (55), and introducing the relations (52). This results in

$$\begin{aligned}
& \frac{Dj_p}{Dt} + j_p (\nabla \cdot v_p) + (j_p \cdot \nabla) v_p - \left( \frac{Q}{\rho_p} - \frac{e_i}{m_i} \right) (\nabla p_{ip} - \nabla \cdot \tau_{ip}) \\
& + \left( \frac{Q}{\rho_p} - \frac{e_e}{m_e} \right) (\nabla p_e - \nabla \cdot \tau_{ep}) + \left( \frac{Q_i^2}{\rho_i} + \frac{Q_e^2}{\rho_e} - \frac{Q^2}{\rho_p} \right) (\underline{E} + v_p \times \underline{B}) \\
& + \left( \frac{e_e}{m_e} + \frac{e_i}{m_i} - \frac{Q}{\rho_p} \right) (j_p \times \underline{B}) + \left( \frac{e_i}{m_i} - \frac{Q}{\rho_p} \right) \underline{P}_{in} + \left( \frac{e_e}{m_e} - \frac{Q}{\rho_p} \right) \underline{P}_{en} \\
& + \left( \frac{e_i}{m_i} - \frac{e_e}{m_e} \right) \underline{P}_{ie} + \frac{Q}{\rho_p} (\omega_i + \omega_e) v_p
\end{aligned} \tag{57}$$

As will be seen subsequently, this equation takes on a more recognizable form upon substitution of appropriate forms of the coupling coefficients  $\underline{P}_{in}$ ,  $\underline{P}_{en}$ , and  $\underline{P}_{ie}$ .

The energy equations for ions and for electrons for the three-temperature model are obtained directly from Eqs. (13), (14), or (15). The form corresponding to Eq. (15) is as follows for ions:

$$\begin{aligned}
& \rho_p \frac{D^p}{Dt} \left[ \frac{\rho_i}{\rho_p} \left( \hat{h}_{ip} + \epsilon_i + \frac{v_p^2}{2} \right) \right] - \rho_p \frac{D^p}{Dt} \left[ \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) \frac{j_p \cdot v_p}{\rho_p} \right] \\
& - \nabla \cdot \left[ \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) j_p \frac{v_p^2}{2} \right] = - \nabla \cdot q_{ip} + \nabla \cdot (v_p \cdot \tau_{ip}) \\
& + \frac{e_i}{m_i} \rho_i \underline{E} \cdot v_p - \frac{e_i}{m_i} \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) \underline{E} \cdot j_p + E_{in} + E_{ie} + \frac{\partial p_{ip}}{\partial t} \\
& - (\omega_i + \omega_e) \left[ \frac{\rho_i}{\rho_p} \left( \hat{h}_{ip} + \frac{v_p^2}{2} + \epsilon_i \right) - \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) \frac{j_p}{\rho_p} \right]
\end{aligned} \tag{58a}$$

For electrons:

$$\begin{aligned}
\rho_p \frac{D^p}{Dt} \left[ \frac{\rho_e}{\rho_p} \left( \hat{h}_{ep} + \epsilon_e + \frac{v_p^2}{2} \right) \right] + \rho_p \frac{D^p}{Dt} \left[ \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) \frac{j_p}{\rho_p} \cdot \underline{v}_p \right] \\
+ \nabla \cdot \left[ \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) j_p \frac{v_p^2}{2} \right] = - \nabla \cdot \underline{q}_{ep} + \nabla \cdot (\underline{v}_p \cdot \underline{\tau}_{ep}) \\
+ \frac{e_e}{m_e} \rho_e (\underline{E} \cdot \underline{v}_p) + \frac{e_e}{m_e} \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) \underline{E} \cdot j_p + E_{en} - E_{ie} \\
+ \frac{\partial p_{ep}}{\partial t} - (\omega_i + \omega_e) \left[ \left( \frac{1}{\frac{e_e}{m_e} - \frac{e_i}{m_i}} \right) \frac{j_p}{\rho_p} \cdot \underline{v}_p \right. \\
\left. + \frac{\rho_e}{\rho_p} \left( \hat{h}_e + \frac{v_p^2}{2} + \epsilon_e \right) \right] \quad (58b)
\end{aligned}$$

For neutrals, the energy equation comes from Eqs. (4a), (4b), or (4c).

From (4c),

$$\begin{aligned}
\rho_n \frac{D^n}{Dt} \left( \hat{h}_n + \epsilon_n + \frac{v_n^2}{2} \right) = - \nabla \cdot \underline{q}_n + \nabla \cdot (\underline{v}_n \cdot \underline{\tau}_n) + \frac{\partial p_n}{\partial t} \\
- \omega_n \left( \hat{h}_n + \epsilon_n + \frac{v_n^2}{2} \right) - E_{en} - E_{in} \quad (58c)
\end{aligned}$$

Forms corresponding to Eqs. (13) and (14) for ions and electrons and Eqs. (4a) and (4b) for neutrals can similarly be obtained. Relations (52c-d) may be used to express the factors  $\rho_i/\rho_p$ ,  $\rho_e/\rho_p$  in terms of  $\rho_p$  and  $Q$ , if desired.

For the two-temperature model, the energy equation for the ion-neutral mixture is obtained by merely adding Eqs. (58a) and (58c); the equation for electrons remains (58b).

For the one-temperature model, the energy equation for the mixture is obtained by adding Eqs. (58a), (58b), and (58c)

$$\begin{aligned}
& \rho_n \frac{D^n}{Dt} \left( \hat{h}_n + \epsilon_n + \frac{v_n^2}{2} \right) + \rho_p \frac{D^p}{Dt} \left[ \frac{\rho_i}{\rho_p} (\hat{h}_{ip} + \epsilon_i) + \frac{\rho_e}{\rho_p} (\hat{h}_{ep} + \epsilon_e) + \frac{v_p^2}{2} \right] \\
& - \nabla \cdot (\underline{q}_p + \underline{q}_n) + \nabla \cdot (\underline{v}_p \cdot \underline{\tau}_p) + \nabla \cdot (\underline{v}_n \cdot \underline{\tau}_n) + \underline{E} \cdot \underline{Qv}_p + \underline{E} \cdot \underline{j}_p \\
& + \frac{\partial p_p}{\partial t} + \frac{\partial p_n}{\partial t} - (\omega_i + \omega_e) \left[ \frac{\rho_i}{\rho_p} (\hat{h}_{ip} + \epsilon_i) + \frac{\rho_e}{\rho_p} (\hat{h}_{ep} + \epsilon_e) + \frac{v_p^2}{2} \right] \\
& - \omega_n \left( \hat{h}_n + \epsilon_n + \frac{v_n^2}{2} \right)
\end{aligned} \tag{59}$$

Similar equations written in terms of either enthalpy or internal energy can also be obtained.

#### B. Approximate Two-Fluid Formulation

If, as in the case of the one-fluid formulation, we introduce the Eqs. (32) through (35) for the form of the momentum coupling terms, and if, further, we assume that  $m_e/m_i \ll 1$ ,  $|e_i m_e/e_e m_i| \ll 1$ , and  $|m_e Q/e_e \rho_p| \ll 1$ ,<sup>†</sup> the equations in the preceding section can be written in the following form.

Continuity:

$$\frac{\partial \rho_p}{\partial t} + \nabla \cdot (\rho_p \underline{v}_p) = \omega_i + \omega_n \tag{60}$$

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<sup>†</sup>It is to be noted that this assumption does not require macroscopic charge neutrality.

$$\frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_n \underline{v}_n) = \omega_n \quad (61)$$

Momentum:

$$\begin{aligned} \rho_p \frac{D^p \underline{v}_p}{Dt} = & - \nabla p_p + \nabla \cdot \underline{\tau}_p + Q(\underline{E} + \underline{v}_p \times \underline{B}) + (\underline{j}_p \times \underline{B}) \\ & - (\omega_i + \omega_e) \underline{v}_p + \eta_{in} (\underline{v}_n - \underline{v}_p) + \underline{\eta}_{en} \cdot (\underline{v}_n - \underline{v}_p) \\ & + \frac{\eta_{in} m_e}{\rho_p e_e} \underline{j}_p + \delta^{**} \left( \frac{m_i}{\rho_p e_i} \right) \underline{\eta}_{en} \cdot \underline{j}_p + \bar{P}_{in} + \bar{P}_{en} \end{aligned} \quad (62)$$

$$\begin{aligned} \rho_n \frac{D^n \underline{v}_n}{Dt} = & - \nabla p_n + (\nabla \cdot \underline{\tau}_n) - \omega_n \underline{v}_n - \eta_{in} (\underline{v}_n - \underline{v}_p) - \underline{\eta}_{en} \cdot (\underline{v}_n - \underline{v}_p) \\ & - \frac{\eta_{in} m_e}{\rho_p e_e} \underline{j}_p - \delta^{**} \left( \frac{m_i}{e_i \rho_p} \right) \underline{\eta}_{en} \cdot \underline{j}_p - \bar{P}_{in} - \bar{P}_{en} \end{aligned} \quad (63)$$

where

$$\delta^{**} \equiv \left( 1 - \frac{Q m_i}{\rho_p e_i} \right)^{-1} \quad (64)$$

Ohm's law (after some rearrangement):

$$\begin{aligned}
\frac{(\underline{j}_p)_{\parallel, \perp}}{\sigma_{\parallel, \perp}} = & (\underline{E} + \underline{v}_p \times \underline{B})_{\parallel, \perp} - \delta^{**} \left( \frac{m_i}{e_i \rho_p} \right) (\underline{j}_p \times \underline{B})_{\parallel, \perp} \\
& + \left( \frac{m_e}{e_e \rho_p} \right) (\nabla \underline{p}_{ip} - \nabla \cdot \underline{\tau}_{ip})_{\parallel, \perp} \\
& + \delta^{**} \left( \frac{m_i}{e_i \rho_p} \right) (\nabla \underline{p}_e - \nabla \cdot \underline{\tau}_{ep})_{\parallel, \perp} \\
& - \delta^{**} \left( \frac{m_i}{e_i} \frac{m_e}{e_e} \right) \frac{Q}{\rho_p^2} (\omega_i + \omega_e) (\underline{v}_p)_{\parallel, \perp} \\
& - \left( \frac{m_e}{e_e \rho_p} \right) \eta_{in} (\underline{v}_n - \underline{v}_p)_{\parallel, \perp} \\
& - \delta^{**} \left( \frac{m_i}{e_i \rho_p} \right) [\underline{\eta}_{en} \cdot (\underline{v}_n - \underline{v}_p)]_{\parallel, \perp} \\
& + \delta^{**} \left( \frac{m_i}{e_i} \frac{m_e}{e_e} \right) \frac{1}{\rho_p} \left[ \frac{D \underline{j}_p}{Dt} + \underline{j}_p (\nabla \cdot \underline{v}_p) + (\underline{j}_p \cdot \nabla) \underline{v}_p \right]_{\parallel, \perp} \\
& - \left( \frac{m_e}{e_e \rho_p} \right) (\bar{P}_{in})_{\parallel, \perp} - \delta^{**} \left( \frac{m_i}{e_i \rho_p} \right) (\bar{P}_{en})_{\parallel, \perp} \\
& + \delta^{**} \frac{m_i}{e_i \rho_p} (\bar{P}_{ie})_{\parallel, \perp}
\end{aligned} \tag{65}$$

where

$$(\sigma_{\parallel, \perp})^{-1} = \frac{1}{\rho_p^2} \left( \frac{m_e}{e_e} \right)^2 \eta_{in} + \frac{(\delta^{**})^2}{\rho_p^2} \left( \frac{m_i}{e_i} \right)^2 \eta_{en\parallel, \perp} + \frac{(\delta^{**})^2}{\rho_p^2} \left( \frac{m_i}{e_i} \right)^2 \eta_{ie\parallel, \perp} \tag{66}$$

Energy (three-temperature model):

$$\begin{aligned}
\rho_p \frac{D^p}{Dt} \left( \hat{h}_{ip} + \epsilon_i + \frac{v_p^2}{2} \right) - \left( \frac{\rho_p m_e}{e_e} \right) \frac{D^p}{Dt} \left( \frac{j_p}{\rho_p} \cdot \underline{v}_p \right) - \left( \frac{m_e}{e_e} \right) \nabla \cdot \left( \underline{j}_p \frac{v_p^2}{2} \right) = \\
- \nabla \cdot \underline{q}_{ip} + \nabla \cdot (\underline{v}_p \cdot \underline{\tau}_{ip}) + \left( \frac{e_i \rho_p}{m_i} \right) (\underline{E} \cdot \underline{v}_p) - \left( \frac{e_i m_e}{e_e m_i} \right) (\underline{E} \cdot \underline{j}_p) \\
+ E_{in} + E_{ie} + \frac{\partial p_{ip}}{\partial t} - (\omega_i + \omega_e) \left[ \left( \hat{h}_{ip} + \epsilon_i + \frac{v_p^2}{2} \right) - \left( \frac{m_e}{e_e \rho_p} \right) \underline{j}_p \right] \quad (67a)
\end{aligned}$$

$$\begin{aligned}
\rho_p \frac{D^p}{Dt} \left[ \frac{1}{\delta^{**}} - \left( \frac{m_e e_i}{m_i e_e} \right) \left( \hat{h}_{ep} + \epsilon_e + \frac{v_p^2}{2} \right) \right] \\
+ \left( \frac{\rho_p m_e}{e_e} \right) \frac{D^p}{Dt} \left( \frac{j_p}{\rho_p} \cdot \underline{v}_p \right) + \left( \frac{m_e}{e_e} \right) \nabla \cdot \left( \underline{j}_p \frac{v_p^2}{2} \right) = \\
- \nabla \cdot \underline{q}_{ep} + \nabla \cdot (\underline{v}_p \cdot \underline{\tau}_{ep}) - \frac{1}{\delta^{**}} \left( \frac{e_i \rho_p}{m_i} \right) (\underline{E} \cdot \underline{v}_p) \\
+ \underline{E} \cdot \underline{j}_p + E_{en} - E_{ie} + \frac{\partial p_{ep}}{\partial t} \\
- (\omega_i + \omega_e) \left[ \frac{1}{\delta^{**}} - \left( \frac{m_e e_i}{m_i e_e} \right) \left( \hat{h}_{ep} + \epsilon_e + \frac{v_p^2}{2} \right) + \left( \frac{m_e}{e_e \rho_p} \right) \underline{j}_p \right] \quad (67b)
\end{aligned}$$



$$\rho_n \frac{D^n}{Dt} \left( \hat{h}_n + \epsilon_n + \frac{v_n^2}{2} \right) = - \nabla \cdot \underline{q}_n + \nabla \cdot (\underline{v}_n \cdot \underline{\tau}_n) + \frac{\partial p_n}{\partial t} - \omega_n \left( \hat{h}_n + \epsilon_n + \frac{v_n^2}{2} \right) - E_{en} - E_{in} \quad (67c)$$

Energy (one-temperature model):

$$\begin{aligned} \rho_n \frac{D^n}{Dt} \left( \hat{h}_n + \epsilon_n + \frac{v_n^2}{2} \right) + \rho_p \frac{D^p}{Dt} \left[ (\hat{h}_{ip} + \epsilon_i) + \frac{1}{\delta^{**}} \left( - \frac{m_e e_i}{m_i e_e} \right) (\hat{h}_{ep} + \epsilon_e) + v_p^2 \right] = \\ - \nabla \cdot (\underline{q}_p + \underline{q}_n) + \nabla \cdot (\underline{v}_p \cdot \underline{\tau}_p) + \nabla \cdot (\underline{v}_n \cdot \underline{\tau}_n) + \underline{E} \cdot \underline{Qv}_p + \underline{E} \cdot \underline{j}_p \\ + \frac{\partial p_p}{\partial t} + \frac{\partial p_n}{\partial t} - (\omega_i + \omega_e) \left[ (\hat{h}_{ip} + \epsilon_i) + \frac{1}{\delta^{**}} \left( - \frac{m_e e_i}{m_i e_e} \right) (\hat{h}_{ep} + \epsilon_e) + v_p^2 \right] - \omega_n \left( \hat{h}_n + \epsilon_n + \frac{v_n^2}{2} \right) \end{aligned} \quad (68)$$

Equations (60) through (66), in the absence of viscous effects and chemical change, assuming charge neutrality and assuming  $\eta_{en}$ ,  $\eta_{in}$  to be scalars, have been thoroughly discussed by Lehnert.<sup>4</sup> In particular, Lehnert assumes  $\eta_{en} = m_e n_e \nu_{en}$ ,  $\eta_{in} = m_i n_i \nu_{in}$ , and  $\eta_{ie} = m_e n_e \nu_{ie}$ , where the  $\nu$ 's are suitably defined collision frequencies.

## NOMENCLATURE

$\underline{B}$	magnetic intensity
$\underline{C}_a$	peculiar velocity of a particle of species $a$ ; $\underline{C}_a = \underline{V}_a - \underline{v}_a$
$\underline{C}_{ap}$	peculiar velocity of a particle of species $a$ relative to mass velocity of plasma: $\underline{C}_{ap} = \underline{V}_a - \underline{v}_p$
$\underline{C}_{at}$	peculiar velocity of a particle of species $a$ relative to mass velocity of mixture; $\underline{C}_{at} = \underline{V}_a - \underline{v}$
$D_{aj}^{(1)}$	binary diffusion coefficient of species $a$ and $j$
$D_a^T$	thermal diffusion coefficient of species $a$
$D^a/Dt$	substantial derivative operator relative to species $a$ ; $D^a/Dt = (\partial/\partial t) + \underline{v}_a \cdot \nabla$
$D/Dt$	substantial derivative operator relative to mixture: $D/Dt = (\partial/\partial t) + \underline{v} \cdot \nabla$
$\underline{E}$	electric field
$E_{aj}$	rate of energy transfer per unit volume from species $j$ to species $a$
$e_a$	charge of a particle of species $a$
$f$	molar fraction of neutral particles; $f = n_n/(n_i + n_n)$
$f_a$	velocity distribution function of species $a$
$\hat{h}_a$	specific perfect gas enthalpy of species $a$ relative to mass velocity of species; $\hat{h}_a = \hat{u}_a + p_a/\rho_a$
$\hat{h}_{ap}$	specific perfect gas enthalpy of species $a$ relative to mass velocity of plasma; $\hat{h}_{ap} = \hat{u}_{ap} + p_{ap}/\rho_a$

# NOMENCLATURE (Continued)

$\hat{h}_{at}$	specific perfect gas enthalpy of species $a$ relative to mass velocity of mixture; $\hat{h}_{at} \equiv \hat{u}_{at} + p_{at}/\rho_a$
$\underline{J}$	total current density; $\underline{J} = \sum_{a=1}^N n_a e_a \underline{v}_a$
$\underline{j}$	conduction current density relative to mixture, $\underline{j} = n_i e_i \underline{w}_i + n_e e_e \underline{w}_e$
$\underline{j}_p$	conduction current density relative to plasma, $\underline{j}_p = n_i e_i \underline{w}_{ip} + n_e e_e \underline{w}_{ep}$
$k$	Boltzmann's constant
$m_a$	mass of particle of species $a$
$N$	number of species present
$n$	total number density of mixture
$n_a$	number density of species $a$
$\underline{P}_{aj}$	rate of momentum transfer per unit volume from species $j$ to species $a$
$p_p$	total plasma pressure $p_p \equiv p_{ip} + p_{ep}$
$p_t$	total pressure of mixture $p_t \equiv \sum_{a=1}^N p_{at}$
$p_a$	partial pressure of species $a$ relative to mass velocity of species; $p_a \equiv 1/3 \rho_a \overline{C}_a^2$
$p_{ap}$	partial pressure of species $a$ relative to mass velocity of plasma; $p_{ap} \equiv 1/3 \rho_a \overline{C}_{ap}^2$

# NOMENCLATURE (Continued)

$p_{at}$	partial pressure of species $a$ relative to mass velocity of mixture; $p_{at} \equiv 1/3 \rho_a \overline{C_{at}^2}$
$Q$	total charge density of mixture; $Q \equiv \sum_a Q_a$
$Q_a$	charge density of species $a$ ; $Q_a = n_a e_a$
$q_t$	energy flux vector of mixture; $q_t \equiv \sum_a q_{at}$
$q_a$	energy flux vector of species $a$ relative to mass velocity of species; $q_a = \rho_a \overline{C_a^2 C_a} + \rho_a \epsilon_a \overline{C_a}$
$q_{ap}$	energy flux vector of species $a$ relative to mass velocity of plasma; $q_{ap} \equiv \rho_a \overline{C_{ap}^2 C_{ap}} + \rho_a \epsilon_a \overline{w_{ap}} = q_a - \overline{w_{ap}} \cdot \overline{T_{ap}}$ $+ \rho_a \overline{w_{ap}} [ \hat{u}_{ap} + (p_{ap}/\rho_a) + \epsilon_a ]$
$q_{at}$	energy flux vector of species $a$ relative to mass velocity of mixture; $q_{at} \equiv \rho_a \overline{C_{at}^2 C_{at}} + \rho_a \epsilon_a \overline{w_a} = q_a - \overline{w_a} \cdot \overline{T_{at}}$ $+ \rho_a \overline{w_a} [ \hat{u}_{at} + (p_{ap}/\rho_a) + \epsilon_a ]$
$T$	temperature of mixture
$T_g$	temperature of ion-neutral mixture
$T_a$	temperature of species $a$ ; $T_a \equiv 1/3 (m_a/k) \overline{C_a^2}$
$t$	time
$\hat{u}_a$	mean translational kinetic energy per unit mass of a particle of species $a$ relative to mass velocity of species; $\hat{u}_a \equiv 1/2 \overline{C_a^2} = 3/2 (k/m_a) T_a$

# NOMENCLATURE (Continued)

$\hat{u}_{ap}$	mean translational kinetic energy per unit mass of a particle of species $a$ relative to mass velocity of plasma; $\hat{u}_{ap} \equiv 1/2 \overline{C_{ap}^2}$
$\hat{u}_{at}$	mean translational kinetic energy per unit mass of a particle of species $a$ relative to mass velocity of mixture; $\hat{u}_{at} \equiv 1/2 \overline{C_{at}^2}$
$\underline{v}_a$	velocity of a particle of species $a$
$\underline{v}$	mass velocity of mixture; $\rho \underline{v} = \sum_a \rho_a \underline{v}_a$
$\underline{v}_A$	ambipolar diffusion velocity; $\underline{v}_A \equiv \underline{v}_p - \underline{v}$
$\underline{v}_p$	mass velocity of plasma; $\rho_p \underline{v}_p = \rho_i \underline{v}_i + \rho_e \underline{v}_e$
$\underline{v}_a$	mass velocity of species $a$ ; $\underline{v}_a = (1/n_a) \int f_a dV_a$
$\underline{w}_a$	diffusion velocity of species $a$ relative to mixture; $\underline{w}_a \equiv \underline{v}_a - \underline{v}$
$\underline{w}_{ap}$	diffusion velocity of species $a$ relative to plasma; $\underline{w}_{ap} \equiv \underline{v}_a - \underline{v}_p$
$\epsilon_a$	energy per unit mass of a particle of species $a$ , excluding translational energy
$\rho$	mass density of mixture; $\rho = \sum_a \rho_a$
$\rho_p$	mass density of plasma; $\rho_p = \rho_i + \rho_e$
$\rho_a$	mass density of species $a$ ; $\rho_a = n_a m_a$
$\underline{\tau}_p$	shear tensor of plasma $\underline{\tau}_p \equiv \underline{\tau}_{ip} + \underline{\tau}_{ep}$
$\underline{\tau}_t$	shear tensor of mixture $\underline{\tau}_t \equiv \sum_a \underline{\tau}_{at}$

# NOMENCLATURE (Continued)

$\underline{\underline{\tau}}_a$	shear tensor of species $a$ relative to mass velocity of species; $\underline{\underline{\tau}}_a \equiv p_a \underline{\underline{I}} - \rho_a \overline{\underline{\underline{C}}_a \underline{\underline{C}}_a}$
$\underline{\underline{\tau}}_{ap}$	shear tensor of species $a$ relative to mass velocity of plasma; $\underline{\underline{\tau}}_{ap} \equiv p_{ap} \underline{\underline{I}} - \rho_a \overline{\underline{\underline{C}}_{ap} \underline{\underline{C}}_{ap}} = p_{ap} \underline{\underline{I}} - p_a \overline{\underline{\underline{C}}_a \underline{\underline{C}}_a} - \rho_a \underline{\underline{w}}_{ap} \underline{\underline{w}}_{ap}$
$\underline{\underline{\tau}}_{at}$	shear tensor of species $a$ relative to mass velocity of mixture; $\underline{\underline{\tau}}_{at} \equiv p_{at} \underline{\underline{I}} - \rho_a \overline{\underline{\underline{C}}_{at} \underline{\underline{C}}_{at}} = p_{at} \underline{\underline{I}} - p_a \overline{\underline{\underline{C}}_a \underline{\underline{C}}_a} - \rho_a \underline{\underline{w}}_a \underline{\underline{w}}_a$
$\omega_a$	mass rate of production per unit volume of species $a$

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UNCLASSIFIED	<p>Aerospace Corporation, El Segundo, California. THE GOVERNING MACROSCOPIC EQUATIONS OF PARTIALLY IONIZED GASES, prepared by D. M. Dix. 25 July 1962. [44]p. (Report TDR-69(2230-04)TN-2; DCAS- TDR-62-133) (Contract AF 04(695)-69) Unclassified report</p> <p>Useful formulations of the macroscopic conservation relations for a three-component mixture composed of ions, electrons, and neutral particles are developed from the basic forms of the equations for the individual species. The development represents primarily an orderly compilation of the work of previous investigators, although the equations developed are somewhat more general than those previously derived. In particular, all inertial and viscous terms are retained, and the temperatures of the individual species are not required to be equal. In addition, treatment of the energy equations is included.</p>
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